Estimation of Multiproduct Models in Economics on the Example of Production Sector of Russian Economy

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Abstract. The model of the real sector of the Russian economy is presented. It allows for the separate description of GDP and its components by expenditure both in constant and in current prices. Unlike standard macroeconomic models, the model proposed considers a set of Trader agents in addition to Producer agent. Traders are based on a set of CES-functions and allow to decompose the statistics available into a set of unobserved components. The Producer is based on a specific production function that performs well for Russian data and works with financial variables, such as credits and bank accounts. In contrary to the standard approach, the model is not linearized to get estimates of model parameters but is estimated directly using a set of nonlinear equations. The optimization is performed numerically and allows to get both series of unobserved model products and their prices and model parameters. The stability of the solution found is checked on simulated data.

Keywords: Macroeconomic modeling · Nonlinear models
Gross Domestic Product (GDP) · Mathematical programming

1 Introduction

A very common approach in modern macroeconomics is to ignore the multiproduct structure of economics and to describe the whole economy as an entity producing one single product (usually it is Gross Domestic Product, GDP). However, such description often lacks both economic sense and forecasting accuracy of resulting models. The first is due to the structure of modern economic activity, where a country's production scopes far beyond goods that can be easily

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accounted for and ranges from physical assets (that are easy to measure) to services and other intangible assets.

The problems with forecasting accuracy are mostly the result of the GDP structure and the way prices are calculated. One of the most common ways to describe the GDP is what is called GDP by expenditure: the decomposition of GDP into consumption expenditures, investment, government consumption, exports, and imports. All these GDP components are initially calculated in current (observable) prices and then deflated to get numbers, that can be compared at different moments of time. This procedure is performed independently for each GDP component, thus deflators of components may differ significantly. Hence, a model with one product has forecasting accuracy that is naturally limited by the difference in deflators of GDP components.

The standard approach in conventional macroeconomics [6,14] is to describe the economy as a set of macroeconomic agents, each of which has its own goals (maximization of expected discounted utility for the consumer, maximization of expected discounted profits for the producer) that interact via demand, supply and prices formation. The economy here produces a single product that is distributed between consumption, investment and other uses.

The interest to multiproduct nature of the economy is growing recently, with a growing number of researchers studying this issue. Speaking about macroeconomics modeling, especially DSGE (Dynamic Stochastic General Equilibrium) modeling, it is a range of models started by [5] and further developed mostly in terms of models studying product turnover (entry and exit), such as [3,10,11]. Producers here are described as entities working in the multiproduct economy, with an infinite, of finite but very large, set of goods produced and used.

The main problem arising here is that multiple goods produced by firms are consequently aggregated by a set of CES (constant elasticity of substitution) functions to some aggregate. CES function, widely used in economics since [7], is a homogeneous of degree 1 function $X = (\sum_{i=1}^{N} X_i^{\rho})^{1/\rho}$, it is widely used in economics as it contains as cases many widely used function, such as linear function, Cobb-Douglas function and Leontief function.

The standard practice of using the same parameters of these aggregating functions for different agents and sets of goods leads to the restriction of attention to de facto one single good again. Prices of all goods and their sets become equal and the analysis performed is restricted by one good with one price in the same fashion as it was described above.

Another common problem with conventional macroeconomic models is the way they are estimated. Due to the high complexity of problems and the resulting equations (rationality conditions of economic agents - solutions of their optimization problems) the standard practice is to log-linearize the model equations around some steady state (equilibrium condition of economics) and estimate the resulting linear equations using econometric techniques. This approach works well for stable periods of economic growth, but it normally cannot explain deviations from the equilibrium state of economics, such as economic shocks, crises, sharp changes in some indicators etc.

The latter problem is addressed in [2,13]. The models here are solved "as is" without linearization and are capable of forecasting during periods different than balanced growth path.

The current paper aims to solve the problems described above. We consider a multiproduct economy in some sense close to [11], but with no restrictions on coefficients of aggregating functions and we solve it directly in a fashion of [2]. Our approach introduces multiple products in a bit different fashion then it is done usually. We consider an economy with a finite number of intermediate products (three in this case, as calculations on real data show that this number is sufficient to get sufficiently high quality of model), where main economic agents (those that represent production – GDP itself and GDP components, such as consumption) work with one aggregated product (each agent, though, works with its own product).

The aggregation of intermediate products to final products and disaggregation of GDP produced into three intermediate products is performed by Traders - specialized agents, that purchase intermediate products, aggregate them to final product using a CES aggregation function and sell the final product directly to agents. The separation of traders and other agents allows us to solve their problems independently, making the solution and estimation process much simpler. The solution of traders' problems yields what is called in [12] multiproduct model decomposition – a methodology of obtaining unobserved intermediate products series from the available GDP statistics, based on a set of equations obtained as a solution of traders' problems.

The agents in this scheme may solve quite standard problems, for example, of maximization of discounted dividends and maximization of discounted utility for Producer and Consumer respectively. The simplicity of their problems allows us to estimate the nonlinear relationships, such as production function or agents' optimality conditions, directly, without linearization. In the same time, we continue to work in the multiproduct environment, thus, we still can model different GDP components as having different prices and model the dynamics of series both in constant and in current prices.

The simple model economy presented in this paper consists of a set of Traders, one per each GDP component (consumption, investment, government expenditure, imports and exports), plus the Trader of final good, and two agents: the Producer, that produces GDP using labor and capital as inputs, and Aggregate Consumer that maximizes specific multiproduct utility function with the possibility to have deposits as financial instrument. The usual practice is to define labor as an exogenous variable, e.g. standard Ramsey-Kass-Koopmans model. In this paper, the consumer model has endogenous consumption and labor, which makes more sense regarding consumer behavior modeling. One of the most common utility function types is CRRA function which is widely used in macroeconomic models, e.g. DSGE models [1]. Consumption component usually follows standard CRRA form, while labor is additive in the utility function. Moreover, labor function form may vary from linear too, for example, CRRA [8,9]. In this paper, labor is included multiplicatively as CRRA function.

2 Traders and Multiproduct Decomposition

2.1 Description of a Typical Trader Agent

The standard Trader agent solves a problem of the following type. It minimizes its expenditures on the purchase of intermediate goods X_a, X_b, X_c at prices p_a, p_b, p_c

$$p_x(t)X(t) = p_a(t)X_a(t) + p_b(t)X_b(t) + p_c(t)X_c(t) \rightarrow \underset{X_a(t),X_b(t),X_c(t)}{min} \tag{1}$$

With the restriction that the total amount of final good $X(t)$ to be sold is fixed and is determined as a CES function of intermediate goods:

$$X(t) = \left(\alpha_a \left(\frac{X_a(t)}{X_a(0)}\right)^{\rho} + \alpha_b \left(\frac{X_b(t)}{X_b(0)}\right)^{\rho} + (1-\alpha_a-\alpha_b)\left(\frac{X_c(t)}{X_c(0)}\right)^{\rho}\right)^{1/\rho} \tag{2}$$

By $X(t)$ we denote here one of the GDP components. So, in the case of the trader of consumer good $X(t)$ is consumption expenditure, for the trader of investment good it is investment expenditure, etc.

Equivalently, the problem can be stated as a maximization of the amount of final good $X(t)$ sold with the restriction that the total expenditure is fixed.

Both problems give the same solution. Denoting

$$\Omega^{aX}(t) = \left(\frac{(1-\alpha_a-\alpha_b)p_a(t)X_a(0)}{\alpha_a p_c(t)X_c(0)}\right)^{\frac{1}{\rho-1}} \tag{3}$$

and

$$\Omega^{bX}(t) = \left(\frac{(1-\alpha_a-\alpha_b)p_b(t)X_b(0)}{\alpha_b p_c(t)X_c(0)}\right)^{\frac{1}{\rho-1}} \tag{4}$$

the solution of a typical trader problem looks as following:

$$X_a(t) = X_a(0)\frac{p_X(t)X(t)\Omega^{aX}(t)}{p_a(t)X_a(0)\Omega^{aX}(t) + p_b(t)X_b(0)\Omega^{bX}(t) + p_c(t)X_c(0)} \tag{5}$$

$$X_b(t) = X_b(0)\frac{p_X(t)X(t)\Omega^{bX}(t)}{p_a(t)X_a(0)\Omega^{aX}(t) + p_b(t)X_b(0)\Omega^{bX}(t) + p_c(t)X_c(0)} \tag{6}$$

$$X_c(t) = X_c(0)\frac{p_X(t)X(t)}{p_a(t)X_a(0)\Omega^{aX}(t) + p_b(t)X_b(0)\Omega^{bX}(t) + p_c(t)X_c(0)} \tag{7}$$

These equations can be used to decompose GDP components into unobserved components if we know the prices $p_a(t), p_b(t), p_c(t)$.

The model estimate for price deflator of each GDP component is

$$\hat{p}_X(t) = \left(\alpha_a \left(\frac{p_a(t)}{p_a(0)}\right)^{\frac{\rho}{\rho-1}} + \alpha_b \left(\frac{p_b(t)}{p_b(0)}\right)^{\frac{\rho}{\rho-1}} + (1-\alpha_a-\alpha_b)\left(\frac{p_c(t)}{p_c(0)}\right)^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}} \tag{8}$$

2.2 The Decomposition Scheme

Given $p_a(t), p_b(t), p_c(t)$, we can calculate model series for price deflators of GDP components using (8) and actual decomposition of GDP components into unobservable intermediate products using (5)–(7). For each moment of time we have 5 equations of type (8) (one for each GDP component) and 3 unknown values of prices $p_a(t), p_b(t), p_c(t)$. Thus, we have more equations then unknown values and we can estimate prices of intermediate goods using model price conditions.

The estimation procedure is organized as follows: we select some initial values for $p_a(t), p_b(t), p_c(t)$, calculate model estimates $\hat{p}_X(t)$ for GDP components' prices using (8) and compute the sum of relative errors, that can be minimized to get estimates of intermediate goods prices.

$$\sum_{t=1}^{T} \sum_{X \in \{C,I,G,Im,Ex\}} \frac{p_X(t) - \hat{p}_X(t)}{p_X(t)} \to min \tag{9}$$

The optimization over more than 200 parameters is performed numerically in R, using the SPG method, originally presented in [4], with R adaptation by [15]. Initial values for parameters are selected randomly, several different sets of initial parameters were tested.

To get more stable estimates, one of the intermediate goods (good c) was fixed to be used only in government spending. Thus, we get a decomposition into three goods, where one of them can be interpreted as a government good.

The results of estimation are presented on Figs. 1, 2 and 3. The accuracy of model estimates is shown for the series with the lowest quality (consumption), the accuracy for other GDP components is even higher. MAE (mean absolute error) for consumption is 0.17 trillion roubles, for consumption deflator is around 3%.

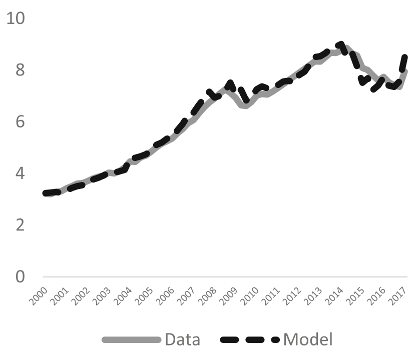

Fig. 1. Accuracy of the model for consumption deflator

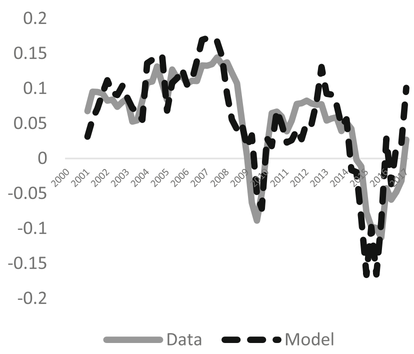

Fig. 2. Accuracy of the model for consumption deflator growth rate

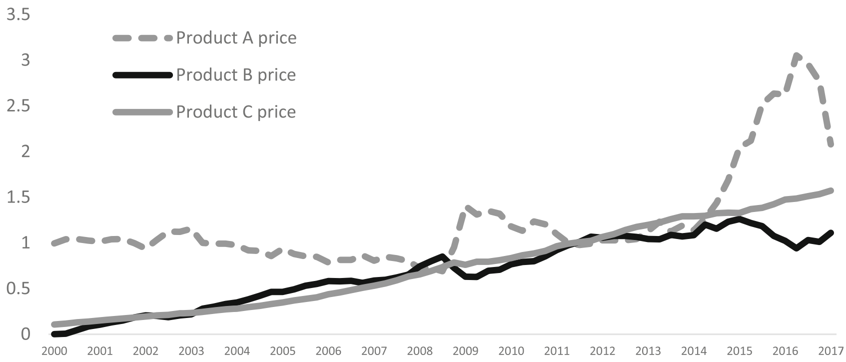

Fig. 3. Model prices of intermediate products

2.3 The Stability of Solution

To check the stability of the solution, we conducted an experiment on simulated data. Prices of intermediate products were generated as random walk processes, GDP components and their price deflators were calculated using (3)–(6) and used to calculate prices of intermediate goods using the decomposition scheme described above. For simplicity of calculations, the stability was checked for the case of two intermediate goods. Initial parameters for optimization were randomly generated, new set for each new iteration.

The stability of the solution found on simulated data is demonstrated in Fig. 4. Real prices are plotted in black (solid and dashed lines), model estimates obtained with different sets of initial parameters are plotted in gray.

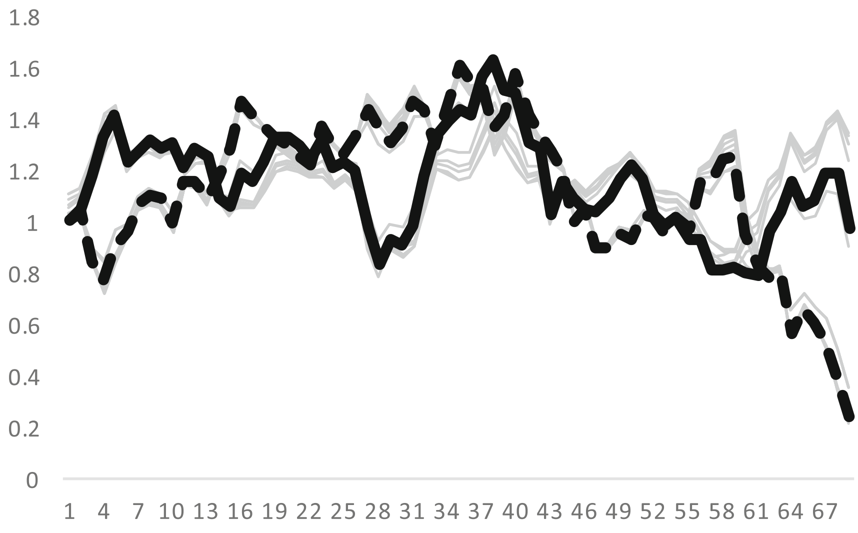

Fig. 4. Stability of decomposition problem solution of simulated data

The convergence to some solution was present in approximately 80% of cases, with MAPE (mean absolute percentage errors) less than 1%. All the solutions found converge to one of the prices (prices of intermediate goods are interchangeable here as there are no links between any of the prices and any other indicator). The behavior of actual prices is replicated with very high quality, the level of series differs a bit, but it is mostly a matter of correct identification of parameters for $t = 0$: different values at the base moment of time can give different levels to otherwise similar trajectories. Apart from that, the solution looks quite stable so we can conclude that the proposed decomposition scheme can be applied to real data.

3 Aggregate Producer

3.1 The Model and Solution

The Producer produces one final good $Y(t)$, using capital $M(t)$ and labor $R(t)$ as inputs. Total investment is divided into capital investment $J_m(t)$ that is used in capital formation, and current investment $J_u(t)$ that is to determine the capital utilization rate. Total investment $J(t) = J_m(t) + J_u(t)$, both investment components are purchased by the producer at price $p_J(t)$. Labor is purchased at price $W_w(t)$.

Production function is a kind of Cobb-Douglas production function with exogenous technological progress:

$$Y_p(t) = Ae^{\gamma t}\left(J_u(t) - u_0 M(t)\right)^{b\alpha}\left(M(t)\right)^{\alpha(1-b)}\left(R(t)\right)^{1-\alpha} \tag{10}$$

where A, γ are some constants that determine the base production level and technological progress respectively, $(J_u(t) - u_0 M(t))$ is the capital utilization rate.

Capital formation is determined by capital investment and amortization:

$$\frac{d}{dt}M(t) = J_m(t) - \delta_{am}(t)M(t) \tag{11}$$

Final product is sold at price $p_Y(t)$ and is taxed at rate $\tau_Y(t)$. Thus, Producer earns $(1 - \tau_Y(t))p_Y(t)Y(t)$.

The producer can take credits, the current amount of credit is denoted as $L(t)$, the credit amount rises when the Producer takes new credits $K(t)$ and lowers when it pays the credit $HL(t)$ with interest $r_l(t)L(t)$. The Producer also has a current account $N(t)$. The producer also pays dividends to its shareholders $Div(t)$. The change in Producer's current account is

$$\begin{aligned}\frac{d}{dt}N(t) = {} & K(t) - HL(t) - r_l(t)L(t) - OC(t) + p_Y(t)Y(t) \\ & - p_J(t)J_u(t) - p_J(t)J_m(t) - W_w(t)R(t) + pS(t) - Div(t)\end{aligned} \tag{12}$$

where $OC(t)$ are other costs, $pS(t)$ are new shares sold by the producer at current prices.

The aim of the Producer is to maximize the utility of the flow of future dividends payed to shareholders, deflated at final good price:

$$\int_0^T U\left(\frac{Div(t)}{p_Y(t)}\right)e^{\Delta t} \tag{13}$$

With respect to restrictions (3)–(5).

The analytical solution of the problem yields the following results:

$$R(t) = \frac{p_Y(t)(\alpha-1)(\tau_Y(t)-1)Y(t)}{W_w(t)} \tag{14}$$

$$\frac{\frac{d}{dt}p_J(t)}{p_J(t)} - r_l(t) = \delta_{am}(t) - \frac{\alpha p_Y(t)(J_u(t)b + u_0 M(t) - J_u(t))(\tau_Y(t)-1)Y(t)}{(J_u(t)-u_0 M(t))M(t)p_J(t)} \tag{15}$$

$$J_u(t) - u_0 M(T) = \frac{\alpha p_Y(t) b(\tau_Y(t)-1)Y(t)}{p_J(t)} \tag{16}$$

Available statistics allows to get series for all prices, $Y(t)$, $J_u(t)$, $J_m(t)$, $M(t)$, $R(t)$, $\tau_Y(t)$. So, we need to estimate only constant parameters of production function.

3.2 Calibration of Model

The model is estimated numerically, with the sum of squared errors in the GDP prediction using production function (10) as a target function that is minimized over production function parameters. The accuracy of resulting model is demonstrated on Figs. 5 and 6.

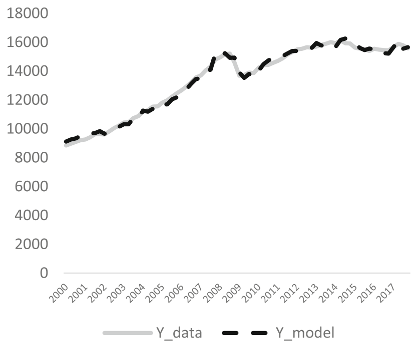

Fig. 5. Accuracy of the model for production function

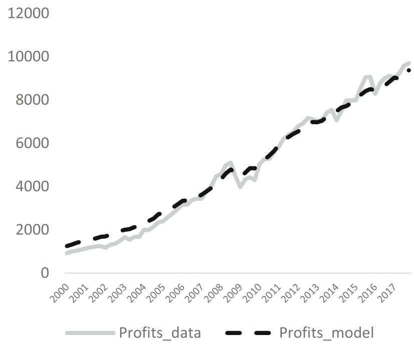

Fig. 6. Accuracy of the model for profits in the economy

Table 1. Aggregate producer model accuracy based on MAE and MAPE coefficients

Variable	MAE	MAPE
GDP Fig. 5	162.66	0.013
Profits Fig. 6	275.24	0.108

The model demonstrates incredibly high accuracy for GDP, with mean absolute percentage error a bit more than 1% and a quite high accuracy for profits predictions. As we can see from the graph, the main problems with profits are due to cyclical fluctuations that start around 2010 and some overestimation of the profits by the model before 2006. The source of the first problem can be in some omitted seasonality (and the problem can be solved by a more thorough work with original data), the second problem is not so serious as we are mostly interested in the more recent data (Table 1).

Other indicators, such as the demand for labor (14), can be calculated using the production function coefficients' estimates and available data.

4 Aggregate Consumer

4.1 The Model and Solution

Aggregate consumer maximizes utility function

$$\int_0^T \frac{C(t)^{1-\beta}}{1-\beta}\frac{R(t)^{1-\alpha}}{1-\alpha}e^{-\delta t}dt \tag{17}$$

choosing both aggregate consumption and labor. In addition, consumer decides the amount of cash $M(t)$ and deposits $S(t)$ taking into account financial balance:

$$\frac{d}{dt}M(t) = \omega(t)R(t) - p(t)C(t) + r_S(t)S(t) - \frac{d}{dt}S(t) - OC(t) \tag{18}$$

considering

$$M(t) \geq 0 \tag{19}$$

$$S(T) \geq \gamma S(0) \tag{20}$$

Wage rate $\omega(t)$, consumption deflator $p(t)$, deposit rate $r_S(t)$ and other incomes $OC(t)$ are known for the whole period [0, T]. Solving given optimization problem the following trajectories for $C(t)$ and $R(t)$ can be obtained:

$$C(t) = C(0)\left[\left(\frac{p(t)}{p(0)}\right)^{\alpha}\left(\frac{\omega(t)}{\omega(0)}\right)^{1-\alpha}e^{-\int_0^t r_S(u)du}e^{\delta t}\right]^{\frac{1}{1-\alpha-\beta}}, \tag{21}$$

$$R(t) = R(0)\left[\left(\frac{p(t)}{p(0)}\right)^{1-\beta}\left(\frac{\omega(t)}{\omega(0)}\right)^{\beta}e^{-\int_0^t r_S(u)du}e^{\delta t}\right]^{\frac{1}{1-\alpha-\beta}} \tag{22}$$

Logarithmizing and then taking derivatives the following equations can be written:

$$C_t = C_{t-1} \frac{1}{1 - \frac{\alpha}{1-\alpha-\beta} \frac{p_t - p_{t-1}}{p_t} - \frac{1-\alpha}{1-\alpha-\beta} \frac{\omega_t - \omega_{t-1}}{\omega_{t-1}} + \frac{1}{1-\alpha-\beta} r_S(t) - \frac{1}{1-\alpha-\beta} \delta} \tag{23}$$

$$R_t = R_{t-1} \frac{1}{1 - \frac{1-\beta}{1-\alpha-\beta} \frac{p_t - p_{t-1}}{p_t} - \frac{\beta}{1-\alpha-\beta} \frac{\omega_t - \omega_{t-1}}{\omega_{t-1}} + \frac{1}{1-\alpha-\beta} r_S(t) - \frac{1}{1-\alpha-\beta} \delta} \tag{24}$$

Assuming (18) it is easy to rewrite discrete equation for $S(t)$:

$$S_t = r_{S_t} S_{t-1} + S_{t-1} + \omega_t R_t - p_t C_t - OC_t \tag{25}$$

4.2 Calibration of Model

Model defines optimal trajectories for consumption $C(t)$ (23), labor $R(t)$ (24) and deposits $S(t)$ (25) with the knowledge of exogenous variables consumption deflator $p(t)$, wage rate $\omega(t)$, deposit rate $r_S(t)$. α, β, δ, $C(0)$, $R(0)$ are calibration parameters that are calculated using minimization of sum of squared errors procedure:

$$\sum_{i=1}^{3} \sum_{t=1}^{T} \frac{(X_i(t) - \hat{X}_i(t))^2}{X_i(T)^2} \tag{26}$$

where $X_i(t) = \{C(t), R(t), S(t)\}$, $\hat{X}_i(t) = \{\hat{C}(t), \hat{R}(t), \hat{S}(t)\}$. $X_i(t)$ stands for statistics and $\hat{X}_i(t)$ stands for model estimated variable. Consumption and labor data is provided by Russian Federation Federal State Statistics Service and deposits data is provided by Central Bank of Russian Federation. Model results are presented by Figs. 7, 8 and 9.

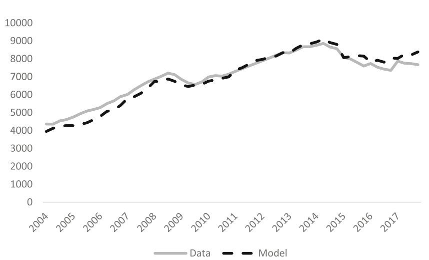

Fig. 7. Accuracy of the model for aggregate consumption

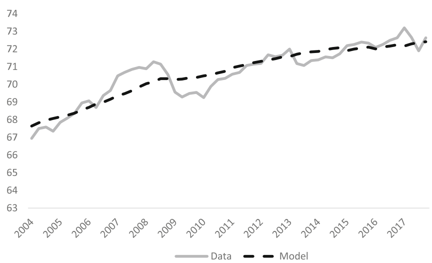

Fig. 8. Accuracy of the model for labor

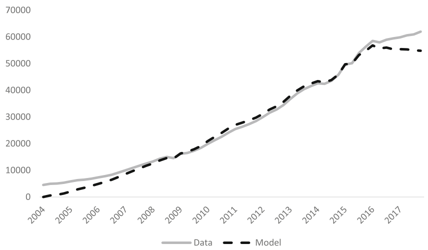

Fig. 9. Accuracy of the model for consumer deposits

Table 2. Aggregate consumer model accuracy based on MAE and MAPE coefficients

Variable	MAE	MAPE
Aggregate consumption Fig. 7	281.36	0.044
Labor Fig. 8	0.496	0.007
Consumer deposits Fig. 9	1867.978	0.153

5 Conclusion

We have presented a methodology for the estimation of multiproduct models in economics. It is based on a set of Trader agents that decompose the original single good produced in the economy into several intermediate products that are unobserved in reality but do constitute in different proportions the GDP and its components. This methodology allows getting the estimates of series of unobserved intermediate products as well as estimates of coefficients of models.

The main advantage of the suggested approach is that it allows taking into account the multiproduct nature of the economy without the necessity to introduce the multiproductivity to the models of main economic agents directly. Using the additional layer of specialized Trader agents that convert multiple intermediate products to the single final product (different for each of the macroeconomic agents) we are able to model multiple products with different prices while still working with conveniently simple single product models of macroeconomic agents (Table 2).

The resulting model is estimated numerically using modern optimization techniques and yields results that are shown to be quite stable and reliable. The accuracy of resulting models is high, and it allows to use them both to explain macroeconomic processes and evaluate policy measures and to make high-quality forecasts of macroeconomic indicators.

References

1. An, S., Schorfheide, F.: Bayesian analysis of DSGE models. Econ. Rev. **26**(2–4), 113–172 (2007)
2. Andreev, M., Pilnik, N., Pospelov, I.G., Vrzheshch, V.P., Masyutin, A.: Intertemporal three-product general equilibrium model of Russian economy. Int. J. Arts Sci. **6**(1), 125–145 (2013)
3. Bilbiie, F.O., Ghironi, F., Melitz, M.J.: Endogenous entry, product variety, and business cycles. J. Polit. Econ. **120**(2), 304–345 (2012)
4. Birgin, E.G., Martinez, J.M., Raydan, M.: Algorithm 813: SPG-software for convex-constrained optimization. ACM Trans. Math. Softw. (TOMS) **27**(3), 340–349 (2001)
5. Chatterjee, S., Cooper, R.: Entry and exit, product variety and the business cycle (No. w4562). National Bureau of Economic Research (1993)
6. Christiano, L., Eichenbaum, M., Rebelo, S.: When Is the government spending multiplier large? J. Polit. Econ. **119**, 78–121 (2011)
7. Dixit, A.K., Stiglitz, J.E.: Monopolistic competition and optimum product diversity. Am. Econ. Rev. **67**(3), 297–308 (1977)
8. Etro, F., Rossi, L.: New-Keynesian Phillips curve with Bertrand competition and endogenous entry. J. Econ. Dyn. Control **51**, 318–340 (2015)
9. Guo, J.T., Krause, A.: Changing social preferences and optimal redistributive taxation. Oxford Econ. Pap. **70**(1), 73–92 (2017)
10. Hamano, M., Zanetti, F.: Endogenous product turnover and macroeconomic dynamics. Rev. Econ. Dyn. **26**, 263–279 (2017)
11. Minniti, A., Turino, F.: Multi-product firms and business cycle dynamics. Eur. Econ. Rev. **57**, 75–97 (2013)
12. Pilnik, N., Pospelov, I., Stankevich, I.: Multiproduct Model Decomposition of Components of Russian GDP. NRU Higher School of Economics. Series WP BRP "Economics/EC". No. WP BRP 111/EC/2015 (2015)
13. Pilnik, N., Radionov, S.: On new approaches to the identification of blocks of general equilibrium models. Proc. MIPT **3**(35), 151–161 (2017)
14. Smets, F., Wouters, R.: Shocks and frictions in US business cycles: a Bayesian DSGE approach. Am. Econ. Rev. **97**(3), 586–606 (2007)
15. Varadhan, R., Gilbert, P.: BB: an R package for solving a large system of nonlinear equations and for optimizing a high-dimensional nonlinear objective function. J. Stat. softw. **32**(4), 1–26 (2009)